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ON TURBULENT SHEAR FLOWS OF VARIABLE DENSITY

John Laufer

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PREFACE

The purpose of this paper is to show that the turbulent field in a variable-density shear layer is equivalent to one of a constant-density layer in which the lateral dimension becomes a random function of time. The results of this study can be applied to the interpretation of laboratory and range data of compressible wakes. This study is part of RAND's work for ARPA on the basic properties of re-entry wakes. The author is Professor of Aerospace Engineering at the University of Southern California and a Consultant to The RAND Corporation.

ABSTRACT

The paper is concerned with the calculation of the mean flow field of free turbulent layers of variable density. It is shown that if the velocity distribution in a particular constant-density flow is known, it is possible to obtain the corresponding variable-density velocity field without the introduction of a compressible turbulent ("eddy") viscosity. This is accomplished by a Dorodnitsyn-Howarth type of transformation applied to the time-dependent rather than to the mean equations of motion, as was done in the past. When the transformed equations are averaged, using Reynolds' method, the incompressible turbulent equations for the mean flow are obtained. These equations can then be handled by conventional methods. It is shown that the predictions obtained by this procedure agree well with experimental results.

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LIST OF SYMBOLS

C_p = specific heat

D = drag or thrust force

d = diameter (of a jet or of a body)

$f\left(\frac{y}{\ell}\right)$ = self-preserving velocity function in a variable-density field

$g\left(\frac{y}{\ell_T}\right)$ = self-preserving temperature function

h = enthalpy

L = characteristic length of the velocity field in a constant-density flow

ℓ = characteristic length of the velocity field in a variable-density flow

ℓ_T = characteristic length of the temperature field in a variable-density flow

M = Mach number

p = pressure

q_k = heat-flux vector

R = gas constant

T = temperature

T_T = total temperature

t = time

U,V,W = velocity components in constant-density flow

u,v,w = velocity components in variable-density flow

X,Y,Z = coordinates in the constant-density field

x,y,z = coordinates in the variable-density field

Γ = mass concentration

γ = ratio of specific heats

θ = momentum diameter

θ_T = energy thickness

ρ = density

$\sigma \equiv \frac{\ell^2}{\ell_T^2}$

τ_{ij} = shearing stress tensor

Subscripts

c = along axis

o = characteristic value ($u_o \equiv u_\infty - u_c$)

1 = jet exit

∞ = conditions outside of shear layer

I. Introduction

The mean velocity field in constant-density, free turbulent shear layers can be quite satisfactorily predicted using a phenomenological approach (especially if the effect of intermittency is included). Such an approach assumes a turbulent exchange coefficient (or turbulent viscosity) that is constant across the shear layer. In a variable-density layer, however, where the Reynolds stresses depend on the density, such an assumption is clearly unsatisfactory. This fact constitutes one of the main difficulties in formulating this problem. The present paper suggests a method that overcomes this difficulty.

In the past, an appropriate transformation has been applied, with reasonable success,[1,2,3,4] to laminar, compressible shear flows, relating them to the incompressible case. This approach has also been used for turbulent flows. In fact, there is a large body of literature in which compressible, turbulent-flow problems are treated by the adoption of a Dorodnitsyn type of scaling -- as used in the laminar case -- and of some rather arbitrary assumptions concerning the compressible,

turbulent-transport terms. The first of these attempts was made by Mager.[5] The only systematic attempt to find a suitable transformation was made by Coles[6] for the turbulent boundary-layer problem. Incidentally, in almost all cases examined by these authors, attention was focused on an incomplete form of the equations of the mean motion.

In formulating the problem at hand, it is appropriate to inquire first whether in fact the application of a relatively simple kinematic transformation of the Dorodnitsyn type is reasonable for turbulent flows. The following observations are pertinent with regard to this point:

(1) Although reliable experiments in variable-density flows are still few in number, there is reasonable evidence that the structure of the turbulent vorticity field is not altered significantly in the presence of density or temperature fluctuations. This suggests that interaction between the vorticity and entropy (or concentration) modes is probably not strong, even in flows of moderate Mach numbers. Indeed, Chu and Kovásznay[7] have shown theoretically that i a homogeneous field these interactions are of second order.

(2) The mean conservation equations indicate that the momentum and energy equations are coupled primarily through the spatial and temporal variations of the density and the transport properties of the gas.[8] For the case of free shear layers, in which the direct effects of the Reynolds and Prandtl numbers are negligible, at least for moderate Mach numbers, the main coupling occurs through the density variation only.

These remarks suggest that an attempt to seek a kinematic transformation that would decouple the momentum and energy equations is a reasonable approach to the problem.

The present paper first rederives the mean momentum and energy equations for a variable-density turbulent flow in a form most convenient for comparison to the constant-density case. In this derivation the turbulent transport terms are generated from the nonlinear convection terms, using a time-averaging procedure. These nonlinear terms, furthermore, are known to be amenable to the Dorodnitsyn-Howarth type of transformation used in laminar flows. It is clear that past difficulties in applying such a transformation to turbulent flows arose from the time-averaged value of these terms. This suggests an alternative approach, in which the Dorodnitsyn-Howarth-Moore transformation is applied to the time-dependent equations of motion and the time-averaging procedure is subsequently carried out with the introduction of certain approximations, a method that allows the recovery of the incompressible Reynolds equations. This result implies that if the mean velocity field in a constant-density flow is known, the corresponding variable-density field can be calculated without the introduction of a hypothesis concerning the compressible turbulent viscosity. We invoke only the classical analogy between the turbulent momentum and heat or mass transfer. Using this procedure, calculations made for several shear flows indicate good agreement with experimental observations.

II. Preliminary Remarks

The conservation equations for a viscous, compressible fluid may be written in the following form:[9]

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho u_j}{\partial x_j} = 0 \tag{1}$$

$$\frac{\partial\rho u_i}{\partial t} + \frac{\partial\rho u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial\tau_{ik}}{\partial x_k} \tag{2}$$

$$\frac{\partial\rho h}{\partial t} + \frac{\partial\rho u_j h}{\partial x_j} = \frac{\partial p}{\partial t} + u_j\frac{\partial p}{\partial x_j} + \tau_{jk}\frac{\partial u_j}{\partial x_k}\frac{\partial q_k}{\partial x_k} \tag{3}$$

We next decompose the various flow quantities into a mean and a fluctuating component, following Reynolds. We will assume that the pressure-fluctuation level p'/p is small compared to that of the velocity. Such an assumption is reasonable in flows in which the amplitude of the velocity fluctuations is small compared to the velocity of sound; this condition is satisfied in flows of moderate Mach number. A more detailed discussion of this point is given by Kistler.[10]

In introducing the velocity and temperature perturbations we choose a mass weighted average, following the suggestion of Morkovin[2] and Favre.[11] Thus,

$$u = \tilde{u} + u', \qquad \rho = \bar{\rho} + \rho' \tag{4}$$

where

$$\tilde{u} \equiv \frac{\overline{\rho u}}{\bar{\rho}} \tag{5}$$

and $\bar{\rho}$ is the conventional time average. It follows that

$$\overline{\rho u'} = 0 \qquad \text{and} \qquad \overline{\rho'} = 0 \tag{6}$$

Similarly, we write

$$T = \tilde{T} + T' \qquad (7)$$

where

$$\tilde{T} \equiv \frac{\overline{\rho T}}{\bar{\rho}} \qquad (8)$$

Substituting Eqs. (4) and (7) into the perfect gas law

$$p = R\rho T \qquad (9)$$

(we restrict the problem to this case) one obtains, after averaging,

$$\bar{p} = R\bar{\rho}\tilde{T} \qquad (10)$$

which retains the form of the perfect gas law. Substituting the perturbations into the conservation relations, averaging, and using boundary-layer approximations, one obtains the following equations:

$$\frac{\partial \bar{\rho}\tilde{u}}{\partial x} + \frac{\partial \bar{\rho}\tilde{v}}{\partial y} = 0 \qquad (11)$$

$$\bar{\rho}\tilde{u}\,\frac{\partial \tilde{u}}{\partial x} + \bar{\rho}\tilde{v}\,\frac{\partial \tilde{u}}{\partial y} = -\frac{\partial \overline{\rho u'v'}}{\partial y} \qquad (12)$$

$$\bar{\rho}\tilde{u}\,\frac{\partial \tilde{T}}{\partial x} + \bar{\rho}\tilde{v}\,\frac{\partial \tilde{T}}{\partial y} = -\frac{\partial \overline{\rho T'v'}}{\partial y} \qquad (13)$$

(The molecular viscosity and heat conductivity terms, as well as the mean pressure gradients, have been neglected.)

The advantages of using mass weighted averages can be readily seen. First, the continuity equation does not have a source term; second, the Reynolds and energy equations retain their forms corresponding

to the incompressible case. It is further seen that the momentum and energy equations are coupled only through the density. Thus, a coordinate transformation involving the density immediately suggests itself. In fact, since the convective terms have the same form as those for laminar flows, many authors have applied to the former a transformation used for the laminar case, as discussed in the Introduction. The turbulent transport terms, however (since they contain a variable density), cause a basic difficulty in applying the transformation.

The main purpose of this paper is to devise a method to overcome this difficulty. Our method takes advantage of the fact that the turbulent transport quantities arise from the nonlinear convective terms, which are amenable to the Dorodnitsyn type of coordinate transformation. This fact immediately suggests that the transformation should be applied not to the time-averaged Reynolds equations, but to the nonstationary conservation Eqs. (1) and (2). The perturbations are subsequently introduced into the transformed equations and the averaging procedure is carried out. The turbulent-transport terms generated by the averaging process are found to be indeed independent of the local density.

For subsequent calculations it is useful to introduce some well-known integral relations of the above equations. By integrating Eq. (11) with respect to y, one obtains the momentum integral relation

$$\frac{d}{dx}\int_{-\infty}^{\infty} \tilde{\rho}\tilde{u}(\tilde{u} - U_{\infty})\, dy = 0 \qquad (14)$$

where U_{∞} is the velocity outside of the shear layer. Integrating with respect to x, we obtain

$$\int_{-\infty}^{\infty} \bar{\rho}\tilde{u}(U_\infty - \tilde{u})\, dy = \text{const.} \equiv D \tag{15}$$

where D corresponds to the wake drag on a body for a wake flow or to the momentum flux through an orifice for a jet flow.

A similar relation is obtained from the thermal energy equation

$$\int_{-\infty}^{\infty} \bar{\rho}\tilde{u}(\tilde{T} - T_\infty)\, dy = \text{const.} = E \tag{16}$$

where C_pE is the excess enthalpy flux in the wake and that at the orifice exit for a jet flow. Since a constant pressure field is assumed, Eq. (16) can be rewritten in the following form using relation (10):

$$T_\infty \int_{-\infty}^{\infty} \bar{\rho}\tilde{u} \left(\frac{\tilde{T}}{T_\infty} - 1\right) dy = T_\infty \int_{-\infty}^{\infty} \bar{\rho}\tilde{u} \left(\frac{\rho_\infty}{\bar{\rho}} - 1\right) dy$$

and therefore

$$T_\infty \int_{-\infty}^{\infty} \tilde{u}(\rho_\infty - \bar{\rho})\, dy = E \tag{17}$$

Another useful integral relation may be obtained for the mean kinetic energy flux. If the momentum equation (12) is multi[illegible]ed by $\tilde{u}$ and integrated with respect to y, one obtains

$$\frac{d}{dx} \int_{-\infty}^{\infty} \bar{\rho}\tilde{u}(\tilde{u} - U_\infty)^2\, dy = 2 \int_{-\infty}^{\infty} (U_\infty - \tilde{u}) \frac{\partial \overline{\rho u' v'}}{\partial y}\, dy \tag{18}$$

III. The Formal Transformation

As discussed in the Introduction, we are seeking a transformation that would eliminate the density variation in the conservation equations

for mass and momentum. To this end we introduce the following transformed coordinates:

$$X = x \tag{19}$$

$$\rho_\infty Y(x,y,z,t) = \int^y \rho(x,y,z,t)\, dy \tag{20}$$

$$Z = z \tag{21}$$

Next, we choose velocity components in the transformed plane (say, the incompressible plane) that satisfy the zero-divergence conditions. This may be accomplished by the following relations

$$U = u \tag{22}$$

$$\rho_\infty V = \rho V + \rho_\infty \left(\frac{\partial Y}{\partial t} + U \frac{\partial Y}{\partial x} + W \frac{\partial Y}{\partial z}\right) \tag{23}$$

$$W = w \tag{24}$$

Substituting these expressions into the continuity equation (1) and the X-component of the momentum equation, one obtains

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} + \frac{\partial W}{\partial Z} = 0 \tag{25}$$

$$\frac{\partial U}{\partial t} + \frac{\partial U^2}{\partial X} + \frac{\partial UV}{\partial Y} + \frac{\partial UW}{\partial Z} = \text{viscous terms} \tag{26}$$

Now we introduce the following perturbations:

$$U(X,Y,Z,t) = \bar{U}(X,Y) + U'[X,Y,Z,t] \tag{27}$$

$$V(X,Y,Z,t) = \overline{V}(X,\overline{Y}) + V'[X,Y,X,t] \tag{28}$$

$$W(X,Y,Z,t) = W'[X,Y,Z,t] \tag{29}$$

$$Y(x,y,z,t) = \overline{Y}(x,y) + Y'(x,y,z,t) \tag{30}$$

Here the bars denote time averages, whence by definition

$$\overline{U'} = \overline{V'} = \overline{W'} = \overline{Y'} \equiv 0 \tag{31}$$

When expressions (27) to (30) are substituted into Eqs. (25) and (26) and averaging is carried out, terms of the form $\partial\Phi/\partial Y$ will arise, where Φ denotes a velocity component or products thereof. Since both Φ and Y are random functions of time, the mean value of their derivative shown above is difficult to determine. Lacking adequate statistical information concerning these functions, we shall assume that

$$\overline{\frac{\partial\Phi}{\partial Y}} \approx \frac{\partial\overline{\Phi}}{\partial\overline{Y}} = \frac{\partial\overline{\Phi}}{\partial\overline{Y}} \tag{32}$$

To clarify the implication of such an approximation,[†] consider the transformed form of Φ, say, $\varphi(x,y,z,t)$; then for the two-dimensional case using Eq. (20),

$$\left(\overline{\frac{\partial\Phi}{\partial Y}}\right) = \left(\overline{\frac{\partial\varphi}{\partial y}\frac{\partial y}{\partial Y}}\right) = \overline{\frac{\partial\varphi}{\partial y}\frac{\rho_\infty}{\rho}} \tag{33}$$

In a single-component system, for instance, $\rho_\infty/\rho = T/T_\infty$; hence

$$\left(\overline{\frac{\partial\Phi}{\partial Y}}\right) = \frac{\partial\overline{\varphi}}{\partial y}\frac{\overline{T}}{T_\infty} + \overline{\frac{\partial\varphi'}{\partial y}\frac{T'}{T_\infty}} \tag{34}$$

[†] This procedure was suggested to the author by Dr. Y. H. Pao.

On the other hand,

$$\overline{\frac{\partial \Phi}{\partial Y}} = \overline{\frac{\partial \varphi}{\partial y} \frac{\partial y}{\partial Y}} = \overline{\frac{\partial \varphi}{\partial y} \frac{T}{T_{\infty}}}$$

$$= \frac{\partial \bar{\varphi}}{\partial y} \frac{\bar{T}}{T_{\infty}} \tag{35}$$

Thus the approximation (32) implies that the correlation between the fluctuations of the temperature and velocity gradients is neglected. Whether this is justifiable can only be established experimentally.

Under these assumptions the time-averaged forms of Eqs. (25) and (26) become, using a boundary-layer type of approximation,

$$\frac{\partial \bar{U}}{\partial X} + \frac{\partial \bar{V}}{\partial \bar{Y}} = 0 \tag{36}$$

$$\frac{\partial \bar{U}^2}{\partial X} + \frac{\partial \bar{U}\bar{V}}{\partial \bar{Y}} = -\frac{\partial}{\partial \bar{Y}} \overline{U'V'} + \text{viscous terms} \tag{37}$$

This, of course, is the incompressible form of the mean flow equations. It is seen, therefore, that under the assumption expressed by Eq. (32), the compressible Reynolds equation may be transformed into the incompressible form for free shear flows with a constant pressure field. This implies that the transformation (19) to (24) provides a means of obtaining the velocity distribution in a variable-density shear flow. According to Eq. (22) the axial component of the velocity profile is in fact identical to that of the corresponding constant-density flow, provided that the lateral coordinates are appropriately stretched in accordance with Eq. (20). To carry out the coordinate transformation, on the other hand, we need to know the density (or temperature) distri-

bution. The latter may be obtained by assuming the classical analogy between momentum and mass (or heat) transfer, thereby introducing a turbulent Prandtl number or a ratio between the characteristic length scales of the two processes. The only undetermined constant in the foregoing method is connected with the virtual origin of the shear flow. Conditions preceding the development of turbulent shear flows are generally complicated, especially if a laminar-to-turbulent transition is involved. However, this problem is beyond the scope of the present paper.

Since many applications involve an axisymmetric geometry, the foregoing analysis has been carried out for this case also. Although the general conclusions are the same, several additional points should be noted.

The coordinate transformation appropriate to an axisymmetric geometry is

$$X = x \tag{19'}$$

$$\frac{1}{2}\rho_\infty Y^2(x,y,\psi,t) = \int^y \rho(x,y,\psi,t)y\,dy \tag{20'}$$

$$\Psi = \psi \tag{21'}$$

where y and ψ are the radial and azimuthal coordinates, respectively. The velocity components in the transformed plane will have the following form:

$$U = u \tag{22'}$$

$$\rho_\infty VY = \rho v y + \rho_\infty Y\left(\frac{\partial Y}{\partial t} + U\frac{\partial Y}{\partial x} + \frac{W}{Y}\frac{\partial Y}{\partial \psi}\right) \tag{23'}$$

$$W = \frac{Y}{y}w \tag{24'}$$

Substituting these expressions into the appropriate forms of the non-stationary compressible-continuity and axial-momentum equations, and following the same steps and assumptions described above, one recovers the incompressible form of the conservation equations.

IV. Applications

In Section III it was shown that by use of the transformations (19) to (24) the flow field in a variable-density turbulent shear flow can be calculated, provided the velocity distribution in the constant-density case is known and an assumption is made concerning the turbulent transport of heat or mass. This assumption can take the form either of an empirical value of the turbulent Prandtl (or Schmidt) number, or of a ratio of two length scales, ℓ and ℓ_T, corresponding to the characteristic scales for the momentum and heat (or mass) transfer, respectively. These conclusions will now be applied to two types of shear flows, the axisymmetric jet and the wake, and the results compared with experimental evidence.

It is well known that for incompressible shear flows, simple integral techniques can predict (except for an undetermined constant) the spreading rate of the turbulent zone and the streamwise variation of the velocity field without the introduction of any phenomenological turbulence theories. This approach will be used here.

The Circular Jet

Experiments indicate that both the mean velocity and mean temperature in a variable-density jet are self-preserving except near the jet exit (see Ref. 20, p. 170). Thus, one may write

$$\tilde{u} = \tilde{u}_c f\left(\frac{y}{\ell}\right) \tag{38}$$

$$T = \bar{T}_\infty + T_o g\left(\frac{y}{\ell_T}\right) \tag{39}$$

Here u_c is the characteristic (or center) velocity and $T_o \equiv T_c - T_\infty$ (the difference between the temperature at the jet's center and the reference temperature just outside of the turbulent zone). The characteristic length ℓ is defined by the following integral:

$$I_1 \equiv \int_o^\infty f\left(\frac{y}{\ell}\right) \frac{y}{\ell}\, d\frac{y}{\ell} = 1 \tag{40}$$

The relation between ℓ and ℓ_T will be discussed later.

The momentum integral corresponding to (15) for a circular jet has the following form:

$$2\pi \int_o^\infty \bar{\rho}\tilde{u}^2 y\, dy = \rho_1 U_1^2 A = \text{const.} \tag{41}$$

The subscript 1 corresponds to the known conditions at the jet exit, and A is the jet area. Applying the transformation (20) and (22), Eq. (41) becomes

$$2\pi\rho_\infty \bar{U}_c^2 L^2 \int_o^\infty F^2\left(\frac{\bar{Y}}{L}\right) \frac{\bar{Y}}{L}\, d\frac{\bar{Y}}{L} = \rho_1 U_1^2 A \tag{42}$$

or

$$\frac{\bar{U}_c^2}{U_1^2} I_2 = \frac{\rho_1}{\rho_\infty}\frac{d^2}{8}\frac{1}{L^2} \equiv \frac{\theta^2}{L^2} \tag{43}$$

where

$$\theta^2 \equiv \frac{\rho_1}{\rho_\infty}\frac{d^2}{8} \tag{44}$$

Note that the momentum diameter θ is the proper reference length of the problem, as pointed out by Kleinstein.[12] In Eq. (42) we used the self-preserving condition

$$\overline{U} = \overline{U}_c F\left(\frac{\overline{Y}}{L}\right) \tag{45}$$

where L is the characteristic length of the constant-density jet and, again,

$$I_1 \equiv \int_o^\infty F\left(\frac{\overline{Y}}{L}\right)\frac{\overline{Y}}{L}\, d\frac{\overline{Y}}{L} = 1 \tag{46}$$

Similarly,

$$I_2 \equiv \int_o^\infty F^2\,\frac{\overline{Y}}{L}\, d\frac{\overline{Y}}{L} \tag{47}$$

The transformed kinetic-energy integral (18) becomes, for a circular jet,

$$\frac{d}{dX}\int_o^\infty \overline{U}^3\overline{Y}\, d\overline{Y} = -2\int_o^\infty \overline{U}\,\frac{\partial\overline{U'V'}\,\overline{Y}}{\partial\overline{Y}}\, d\overline{Y} \tag{48}$$

Assuming now that the Reynolds stresses also follow a self-preserving distribution, so that

$$\overline{U'V'} = U_c^2 G\left(\frac{\overline{Y}}{L}\right) \tag{49}$$

Equation (48) simplifies to

$$\frac{d}{dX}\,\overline{U}_c^3 L^2 I_3 = -2\,\overline{U}_c^3 L I_G \tag{50}$$

where

$$I_3 \equiv \int_0^\infty F^3\,\frac{\overline{Y}}{L}\,d\frac{\overline{Y}}{L} \quad \text{and} \quad I_G \equiv \int_0^\infty \left(\frac{\overline{Y}}{L}\,G\right)'\,d\frac{\overline{Y}}{L} \tag{51}$$

With Eq. (43), Eq. (50) further simplifies to

$$\frac{1}{\overline{U}_c^2}\,\frac{d\overline{U}_c}{dX} = -2\,\frac{I_G\sqrt{I_2}}{I_3}\,\frac{1}{U_1\theta} \tag{52}$$

which becomes, after integration,

$$\frac{U_1}{\overline{U}_c} = 2\,\frac{I_G\sqrt{I_2}}{I_3}\,\frac{X - X_o}{\theta} \tag{53}$$

Substituting Eq. (53) into Eq. (43), we obtain

$$\frac{L}{\theta} = 2\,\frac{I_G}{I_3}\,\frac{X - X_o}{\theta} \tag{54}$$

From measurements of constant-density jets, one finds that

$$2\,\frac{I_G\sqrt{I_2}}{I_3} = 0.052^{\dagger} \tag{55}$$

---

†This value corresponds to one given by Hinze,[13] based on the experiments of Van der Hegge Zijnen,[14] if the fact that $\rho_1/\rho_\infty = .91$ for these experiments is taken into account. It is also consistent with Corrsin and Uberoi's measurements, where $\rho_1/\rho_\infty = .95$.[15]

The value of X_o depends on the flow conditions at the nozzle exit, as discussed in Section III. In low-speed flows, however, this value is small compared to the value of X at which self-preservation begins, and can generally be neglected. A good approximation to the values of the definite integrals I_2 and I_3 may be obtained by assuming a Gaussian distribution for F(Y/L); then $I_2 = 1/2$ and $I_3 = 1/3$. Such an assumption is of course unnecessary if one solves the equations of motion directly using a phenomenological approach. Thus, the solution for the incompressible problem has the form

$$\frac{U_1}{\bar{U}_c} = 0.052 \frac{X - X_o}{\theta} \tag{56}$$

$$\frac{L}{\theta} = 0.073 \frac{X - X_o}{\theta} \tag{57}$$

for values of X sufficiently large that self-preservation holds. Using now the transformation (19) and (22), we can rewrite Eq. (56) for a variable-density jet as

$$\frac{U_1}{\bar{u}_c} = 0.052 \frac{x - x_o}{\theta} \tag{58}$$

The characteristic length ℓ is related to L through the coordinate stretching given by Eq. (20). This, in turn, depends on the density variation. To obtain the density (or temperature) distribution, we need some knowledge of the turbulent heat-transfer mechanism. If we assume that the mechanism is the same as that of the momentum transfer, then the governing Eqs. (12) and (13) have the same form and $g(y/\ell)$ =

$f(y/\ell)$ for similar boundary conditions. Experiments with conventional averaging methods show, however, that the characteristic length for the normalized temperature distribution is larger than that for the velocity. Thus, letting $\sigma \equiv \ell^2/\ell_T^2$,

$$g\left(\frac{y\sqrt{\sigma}}{\ell}\right) = f\left(\frac{y}{\ell}\right) \tag{59}$$

Assuming a Gaussian distribution, this becomes

$$g\left(\frac{y}{\ell}\right) = f^{\sigma}\left(\frac{y}{\ell}\right) \tag{60}$$

From the energy integral (16) applied to the axisymmetric case we obtain

$$2\pi \int_0^\infty \bar{\rho}\tilde{u}(\tilde{T} - T_\infty) y \, dy = \rho_1 U_1 (T_1 - T_\infty) A = \text{const.} \tag{61}$$

or, using Eqs. (38), (39), and (60),

$$\frac{1}{\rho_\infty} \frac{\tilde{u}_c}{U_1} \frac{\tilde{T}_o}{T_\infty} \int_0^\infty \bar{\rho} f^{\sigma+1} y \, dy = \frac{T_1 - T_\infty}{T_\infty} \frac{\rho_1}{\rho_\infty} \frac{d^2}{8} \equiv \frac{\theta_T^2}{\theta^2} \theta^2 \tag{62}$$

$$\frac{\theta_T^2}{\theta^2} \equiv \frac{T_1 - T_\infty}{T_\infty} \tag{63}$$

where θ_T^2 is the "energy thickness" of the jet. Applying the transformation (24) to the integral in Eq. (62), we obtain

$$\int_0^\infty \bar{\rho} f^{\sigma+1} y \, dy = L^2 \rho_\infty \int_0^\infty F^{\sigma+1} \frac{Y}{L} \, d\frac{Y}{L} = \frac{L^2 \rho_\infty}{\sigma + 1} \tag{64}$$

Therefore Eq. (62) becomes

$$\frac{\tilde{T}_o}{T_\infty} = \frac{\theta_T^2}{\theta^2}\frac{\theta^2}{L^2}\frac{U_1}{\tilde{u}_c}(\sigma + 1) \tag{65}$$

or, using Eqs. (57) and (58),

$$\frac{\tilde{T}_c - T_\infty}{T_\infty} \equiv \frac{\tilde{T}_o}{T_\infty} = 9.7\ (\sigma + 1)\ \frac{\theta_T^2}{\theta^2}\frac{\theta}{x - x_o} \tag{66}$$

A similar expression can be obtained for the total temperature decay and concentration decay, since both are governed by an integral equation similar to Eq. (17). Thus

$$\frac{\tilde{T}_{T_c} - T_{T_\infty}}{T_{T_\infty}} = 9.7\ (\sigma + 1)\ \frac{T_{T_1} - T_{T_\infty}}{T_{T_\infty}}\frac{\theta}{x - x_o} \tag{67}$$

and for the mass concentration at the jet's center, we have

$$\Gamma_c = 9.7\ (\sigma + 1)\ \Gamma_1 \frac{\theta}{X - X_o} \tag{68}$$

where Γ_1 is the mass concentration at the orifice exit (the concentration is taken as zero in the ambient field).

Finally, in order to obtain the spreading rate of the jet, we use the alternate form of the energy integral [Eq. (17)] applied to the axisymmetrical case:

$$2\pi T_\infty \int_0^\infty \tilde{u}(\rho_\infty - \bar{\rho})y\ dy = \rho_1 U_1 (T_1 - T_\infty) A \tag{69}$$

or

$$\ell^2 \frac{\tilde{u}_c}{U_1} \int_o^\infty f \frac{Y}{\ell} d\frac{Y}{\ell} - L^2 \frac{\tilde{u}_c}{U_1} \int_o^\infty F \frac{Y}{L} d\frac{Y}{L} = \theta_T^2 \tag{70}$$

But the value of the integrals is unity by definition; therefore

$$\frac{\ell^2}{L^2} - 1 = \frac{\theta_T^2}{\theta^2} \frac{\theta^2}{L^2} \frac{U_1}{\tilde{u}_c} \tag{71}$$

Comparing Eq. (71) to Eq. (65), we obtain

$$\frac{\ell^2}{L^2} - 1 = \frac{1}{\sigma + 1} \frac{\tilde{T}_c}{T_\infty} \tag{72}$$

$$= 9.7 \frac{\theta_T^2}{\theta^2} \frac{\theta}{x - x_o} \tag{73}$$

or, using Eq. (57),

$$\frac{\ell^2}{\theta^2} = 5.3 \times 10^{-3} \left(\frac{x - x_o}{\theta}\right)^2 + 5.2 \times 10^{-2} \frac{\theta_T^2}{\theta^2} \frac{x - x_o}{\theta} \tag{74}$$

It is seen in Eq. (74) that a variable-density jet does not spread linearly. However, unless the "energy thickness" is large, the nonlinear term is negligible in the self-preserving region of the flow.

Equations (58), (66), and (72) completely determine the flow field. The only "adjustable" constant is x_o, which could be estimated by considering the mixing zone and the potential cone near the nozzle exit. Instead, however, we shall choose the value of x_o, using Eq. (72), that gives the best agreement with experimental results. Equations (58) and

(66) can then be unambiguously compared to measurements. In all calculations we use $\sigma = 0.64$, the value obtained from the measurements of Corrsin and Uberoi.[15]

In Fig. 1 the measurements of several investigators are shown. The supersonic jet experiments of Eggers[16] and Johannesen[17] extend well beyond the self-preserving region of the jet, and the virtual origin was easily determined. Only two sets of Warren's[18] data are reproduced; his other subsonic measurements show similar trends, whereas his supersonic ones do not extend far enough in the self-preserving region to be usable. The selected values of x_o/θ are indicated on the figure; the origin of the abscissa was shifted for clarity.

Figure 2 shows the center velocity decay. The solid line corresponds to Eq. (61). It is seen that the agreement with the experiments is quite satisfactory.

In Fig. 3 the center-temperature defect is plotted against the axial-distance parameter $\theta^2 x - x_o/\theta_T^2\theta$. The solid line corresponds to Eqs. (66), (67), and (68); in all cases $\sigma = 0.64$. Also shown are the static temperature measurements of Corrsin and Uberoi,[15] the total temperature data of Warren[18] and the results of Keagy and Weller[19] obtained in a helium jet. (Eggers and Johannesen did not take temperature data.) Although these experiments are comparatively crude (the ratio of probe to jet width is large), they are of interest because of the large density ratio ρ_∞/ρ_1. The agreement between the analysis and the measurements is again satisfactory.

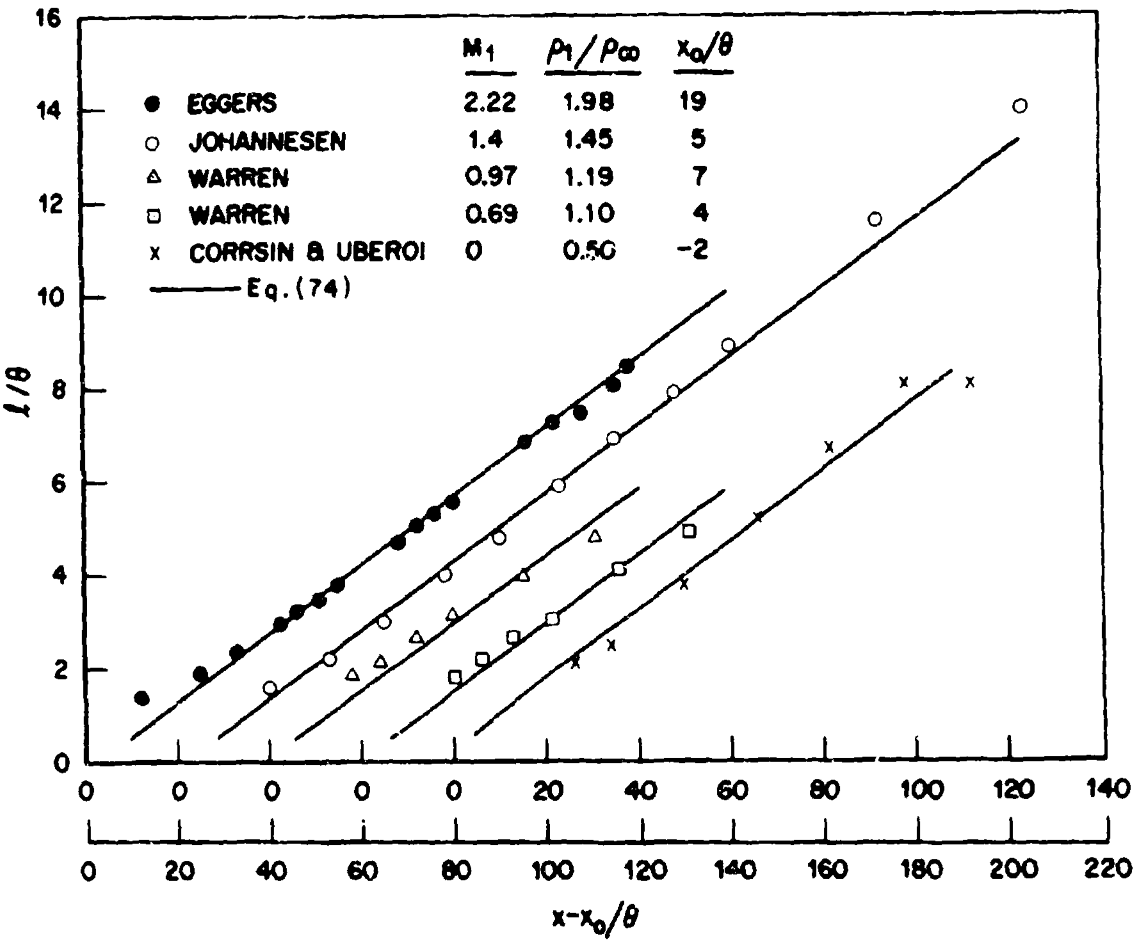

Fig. 1—Jet width

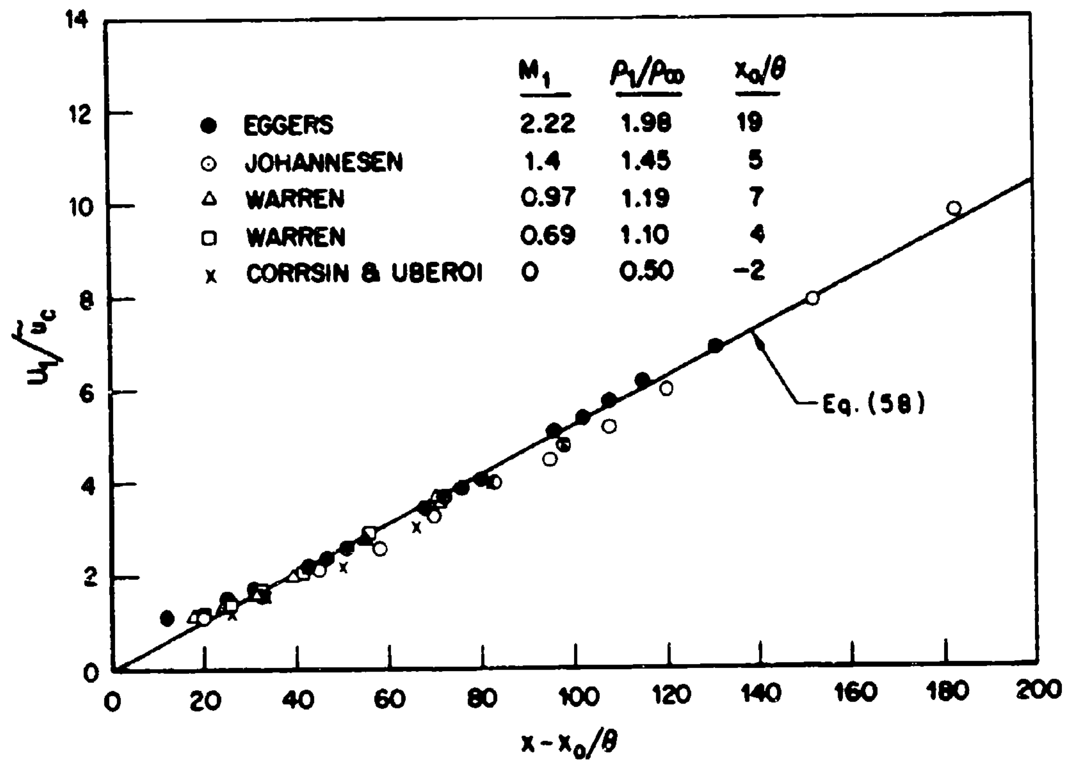

Fig. 2—Center velocity in the circular jet

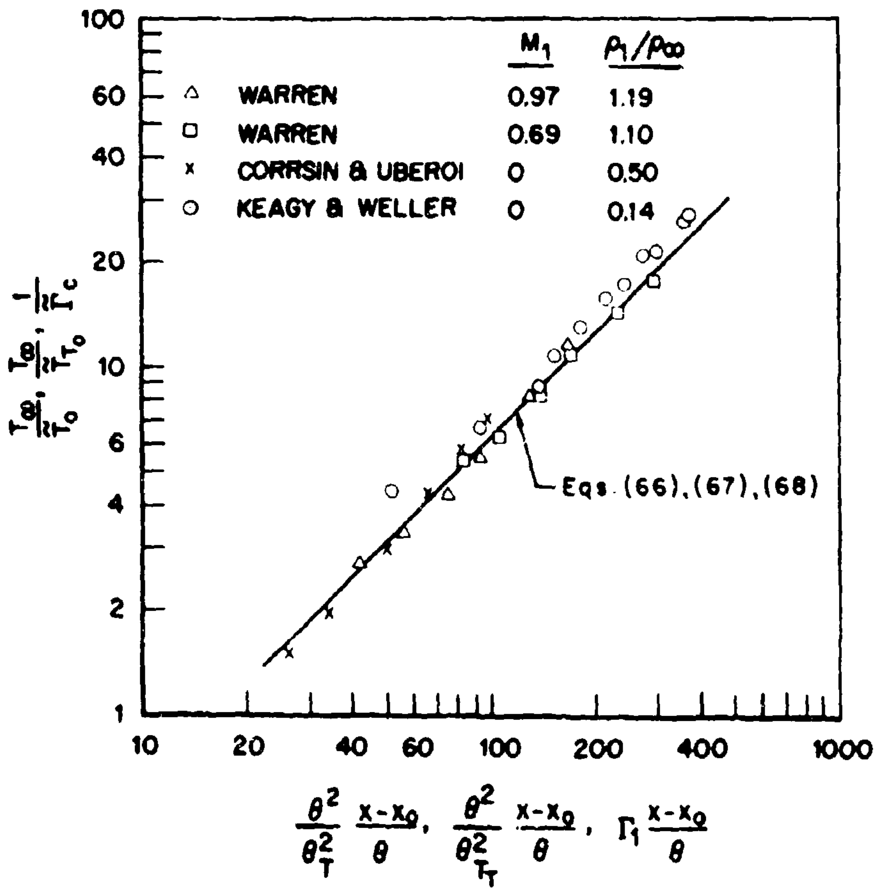

Fig. 3—Center temperature in the circular jet

The Axisymmetric Wake

It is well known that for wake-like flows, self-preservation occurs only if the characteristic velocity $\tilde{u}_o$ (the centerline velocity defect) is small compared to the free-stream velocity. Thus,

$$\tilde{u} = u_\infty - \tilde{u}_o f\left(\frac{y}{\ell}\right) \qquad \text{for} \qquad \frac{\tilde{u}_o}{u_\infty} \ll 1 \tag{75}$$

Under this assumption the temperature distribution is also self-similar, having the form given by Eq. (39).

In the constant-density case, Townsend has already shown[20] that the velocity defect and wake spreading are given by the following equations, neglecting terms of the order $(\overline{U}_o/U_\infty)^2$:

$$\frac{U_\infty}{\overline{U}_o} = \left(\frac{3I_G\sqrt{I_1}}{I_2}\right)^{2/3}\left(\frac{X - X_o}{\theta}\right)^{2/3} \tag{76}$$

$$\frac{L}{\theta} = \left(\frac{3I_G}{I_1 I_2}\right)^{1/3}\left(\frac{X - X_o}{\theta}\right)^{1/3} \tag{77}$$

Here the reference length θ is defined as

$$\theta^2 \equiv \frac{C_D d^2}{16} \tag{78}$$

C_D being the wake drag coefficient of the body. The integrals I_1, I_2, and I_G are given by Eqs. (40), (47), and (51).

There are very few experimental results available for the incompressible, axisymmetric wake. It is thus difficult to obtain a reliable

value for the constants in Eqs. (76) and (77). One observation, however, should be made using the results. According to Reichardt,[21] these constants differ with the shape of the body considered. For instance, the spreading rate of a sphere is higher than that of a cylindrical body (with its axis in the flow direction). One should therefore be careful in comparing constant- and variable-density wake flows.

For the case of a cylindrical body, Hall and Hislop[22] obtained experimentally the value[20]

$$\frac{I_2}{I_G\sqrt{I_1}} = 14.1 \tag{79}$$

Unfortunately, since their measurements have been carried out only to about seventeen diameters downstream of the body, where self-preservation is not yet completely achieved, the above value might be somewhat high. Using the value in Eq. (79) and with $I_1 \equiv 1$, Eqs. (76) and (77) become

$$\frac{U_\infty}{\bar{U}_o} = .36\left(\frac{X - X_o}{\theta}\right)^{2/3} \tag{80}$$

$$\frac{L}{\theta} = .60\left(\frac{X - X_o}{\theta}\right)^{1/3} \tag{81}$$

For the variable-density wake, Eq. (80) retains its form, and the decay of the characteristic temperature can again be calculated from the energy integral. Accordingly,

$$2\pi \int \widetilde{\rho u}(\tilde{T} - T_\infty)y\, dy = E \equiv 2\pi\theta_T^2\rho_\infty U_\infty T_\infty \tag{82}$$

By using arguments similar to those for the jet problem, we may write

$$\frac{\tilde{T} - T_\infty}{\tilde{T}_o} = \left(\frac{u_\infty - \tilde{u}}{\tilde{u}_o}\right)^\sigma \tag{83}$$

so that Eq. (82) becomes

$$\tilde{T}_o L^2 \int_o^\infty (U_\infty - \tilde{U}_o F) F^\sigma \frac{Y}{L}\, d\frac{Y}{L} = \theta_T^2 U_\infty T_\infty \tag{84}$$

This simplifies to

$$\frac{\tilde{T}_o}{T_\infty} = \sigma \frac{\theta_T^2}{\theta^2} \frac{\theta^2}{L^2} \frac{1}{1 - \frac{\sigma}{\sigma + 1} \frac{\tilde{u}_o}{u_\infty}} \tag{85}$$

where again a Gaussian distribution has been assumed for the velocity profile.

To express the "energy diameter" in terms of free-stream conditions, one assumes that all the kinetic energy loss in the wake, as indicated by the momentum defect, is transformed into heat. Thus, using Eqs. (15), (78), and (82), we obtain

$$DU_\infty = E \tag{86}$$

or

$$2\pi\rho_\infty U_\infty^3 \theta^2 = 2\pi\rho_\infty U_\infty C_p T_\infty \theta_T^2 \tag{87}$$

This gives

$$\frac{\theta_T^2}{\theta^2} = (\gamma - 1)M_\infty^2 \tag{88}$$

Substituting this expression into Eq. (85) and using Eqs. (80) and (81), one obtains

$$\frac{\tilde{T}_o}{T_\infty} = 2.8\sigma(\gamma - 1)M_\infty^2\left(\frac{x - x_o}{\theta}\right)^{-2/3} \frac{1}{1 - \frac{2.8\,\sigma}{1 + \sigma}\left(\frac{x - x_o}{\theta}\right)^{-2/3}} \tag{89}$$

From the axisymmetric form of Eq. (17) we again obtain the width of the jet:

$$2\pi T_\infty U_\infty \int_0^\infty (\rho_\infty - \bar{\rho})y\,dy - 2\pi T_\infty \rho_\infty \tilde{u}_o \int_0^\infty fy\,dy$$

$$+ 2\pi T_\infty \tilde{u}_o \int_0^\infty f\bar{\rho}y\,dy = 2\pi\theta_T^2 \rho_\infty U_\infty T_\infty \tag{90}$$

But since

$$\rho_\infty - \bar{\rho} = \bar{\rho}\,\frac{\tilde{T}_o}{T_\infty} f^\sigma \tag{91}$$

Equation (90) becomes

$$\frac{1}{\sigma}\frac{\tilde{T}_o}{T_\infty} L^2 - \frac{\tilde{u}_o}{U_\infty}\ell^2 + \frac{\tilde{u}_o}{U_\infty} L^2 = \theta_T^2 \tag{92}$$

or, using Eq. (85), one recovers Eq. (72) for the jet; that is,

$$\frac{\ell^2}{L^2} - 1 = \frac{1}{\sigma + 1}\frac{\tilde{T}_o}{T_\infty} \tag{93}$$

With Eqs. (80), (81), and (85) this finally becomes

$$\frac{\ell^2}{\theta^2} = .36\left(\frac{x - x_o}{\theta}\right)^{2/3} + \frac{\sigma}{\sigma + 1}(\gamma - 1)M_\infty^2 \frac{1}{1 - \frac{2.8\,\sigma}{\sigma + 1}\left(\frac{x - x_o}{\theta}\right)^{-2/3}} \tag{94}$$

These results are compared to the measurements of Demetriades[23] in Fig. 4. Since Demetriades presents his data in terms of $x - x_o$, Eqs. (80), (89), and (93) can be directly related to the measurements. Again $\sigma = 0.64$ was used for the computations. The agreement between the present analysis and the experiments is acceptable.

V. Conclusion

The analysis carried out in this paper indicates that with the use of a suitable transformation it is possible to predict with reasonable accuracy the mean velocity and density (or temperature) field in a variable-density shear layer, once the velocity distribution in the corresponding constant density layer is known.

Two explicit results follow from the transformation:

(1) The characteristic velocity of a self-similar shear layer is a function of the nondimensional coordinate x/θ only, irrespective of the density variation in the flow. Here θ is a reference length defined by the momentum integral equation.

(2) The ratio of the width of a variable-density layer to that of the corresponding constant-density layer can be expressed explicitly in terms of the local maximum temperature (or concentration) difference. For axially symmetric geometries, this has the form

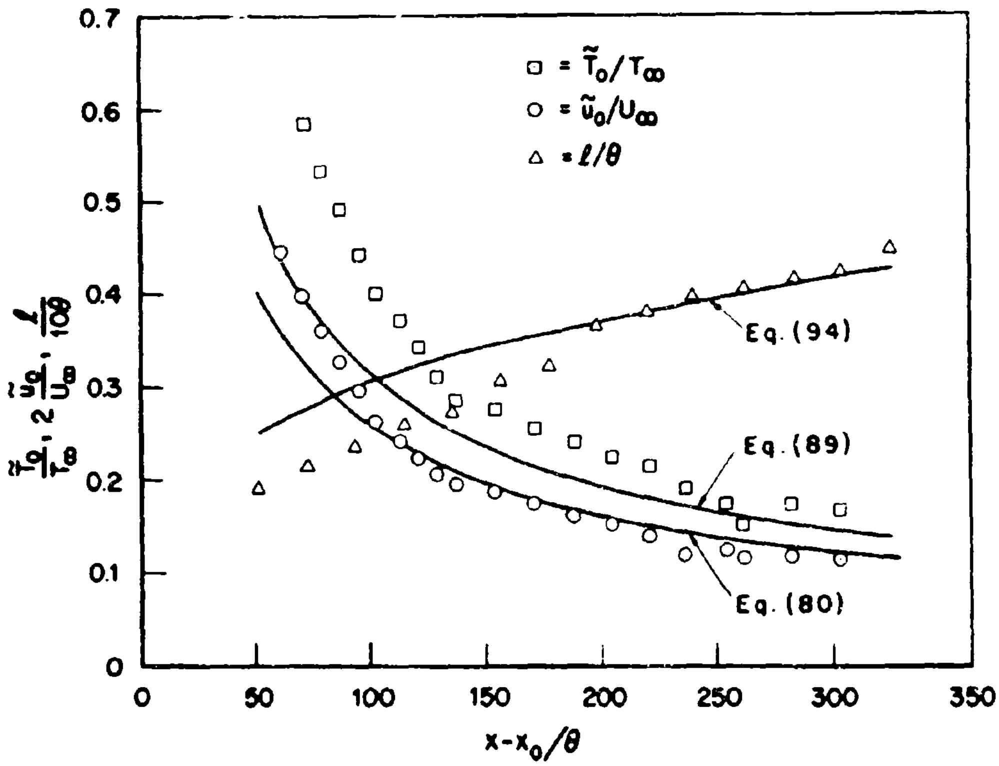

Fig. 4—Axisymmetric wake. Comparison with measurements of Demetriades. (M_∞ = 3.0)

$$\frac{\ell^2}{L^2} = 1 + \frac{1}{\sigma + 1} \frac{\tilde{T}_o}{T_\infty}$$

There are several obvious limitations of the method:

(1) A constant pressure field is assumed. This assumption might be relaxed -- at least for certain types of pressure gradients -- if a more sophisticated transformation were used.

(2) The early stages of the shear-layer development cannot be treated, especially if a laminar-to-turbulent transition occurs. One is then forced to assume a "virtual origin" for the shear layer.

On the other hand, the proposed method opens new possibilities for the analysis of compressible, turbulent boundary layers.

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10. ABSTRACT

A study of the mean flow field of free turbulent layers of variable density showing that if the velocity distribution in a particular constant-density flow is known, it is possible to obtain the corresponding variable-density velocity field without the introduction of a compressible turbulent viscosity. This is accomplished by a Dorodnitsyn-Howarth type of transformation applied to the time-dependent equations of motion rather than to the mean equations of motion, as has been done previously. When the transformed equations are averaged, using Reynolds' method, the incompressible turbulent equations for the mean flow are obtained. These equations can then be handled by conventional methods. Predictions obtained by this procedure agree well with experimental results.

11. KEY WORDS

Aerodynamics
Fluid mechanics
Physics